JOURNAL OF APPROXIMATION THEORY 64, 271–280 (1991)

On Certain Configurations of Points in $\mathbb{R}^n$ Which Are Unisolvent for Polynomial Interpolation

L. Bos

Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4

Communicated by Charles K. Chui

Received September 21, 1989; revised May 22, 1990

We describe how points may be placed on collections of algebraic varieties so that the resulting system is unisolvent for polynomial interpolation. We also give formulas for the corresponding Vandermonde determinants. © 1991 Academic Press, Inc.

1. Polynomial Interpolation in $\mathbb{R}^n$

We will denote the space of polynomials in n real variables by $\mathscr{P}$. $\mathscr{P}_d \subset \mathscr{P}$ will be those of total degree at most d. Now suppose that $E \subset \mathbb{R}^n$ is some set. Let $I(E) := \{p \in \mathscr{P} \mid p \text{ is identically zero on } E\}$. Then, the polynomials of degree d, when restricted to E, form a certain vector space, $\mathscr{P}_d(E)$ say, which we may identify with the factor space $\mathscr{P}_d/I(E)$. For instance if E is the unit circle in $\mathbb{R}^2$, then $\mathscr{P}_d(E)$ is the space of trigonometric polynomials of degree d. Of course $E = \mathbb{R}^n$, or even if E contains some non-empty open set, gives all polynomials of degree d.

Now $\mathscr{P}_d(E)$ has a dimension which we denote by $N_d(E)$. The polynomial interpolation problem is then, given $N_d(E)$ points $x_\alpha \in E$ and $N_d(E)$ function values f_α, to find a $p \in \mathscr{P}_d(E)$ such that $p(x_\alpha) = f_\alpha$, all α. We will say that $X := \{x_1, ..., x_{N_d(E)}\} \subset E$ is unisolvent if the interpolation problem has a unique solution for all given sets of function values.

In studying unisolvency it is useful to introduce the (generalized) Vandermonde determinant. Pick a basis $\{q_1, ..., q_{N_d(E)}\}$ for $\mathscr{P}_d(E)$. Then for X as above,

$$VDM_E^d(X) := \det[q_\alpha(x_\beta)]_{1 \leqslant \alpha, \beta \leqslant N_d(E)}.$$

0021-9045/91 \$3.00

Although this determinant does depend on the basis which is used, it is not difficult to see that the determinants for two different bases differ by a factor of the determinant of the basis transition matrix. Thus for questions of unisolvency, the choice of basis is not important.

Now, it is also not difficult to see that X is unisolvent iff $VDM_E^d(X) \neq 0$. Equivalently, X is unisolvent iff there is no $p \in \mathscr{P}_d(E)$ such that $p(x)=0$, $\forall x \in X$. In one variable, for $E=[a, b]$, it is well known that X is unisolvent as long as the points are distinct. However, these questions are much more difficult in several variables. It is the purpose of this note to give some configurations of points for which it is not too difficult to show unisolvency. These are described in Section 3. In Section 2, as a preliminary, we discuss the special case when E is an algebraic variety. In Section 4 we discuss the computation of Lagrange polynomials and the corresponding Vandermonde determinants. An illustrative example in $\mathbb{R}^2$ is presented in Section 5. These results generalize some of the results of Chung and Yao [3] and Gasca and Maeztu [4], where the authors consider Lagrange interpolation at points constrained to lie on hyperplanes. Further, Chui and Lai [2] give a formula for the Vandermonde determinant in this special case. In [4], the more general problem of Hermite interpolation is also treated but this will not be considered here. Some of these results appeared in a less general form in [1].

2. The Case of E an Algebraic Variety

Suppose now that E is a real algebraic variety such that its ideal, $I(E)$, is principal; i.e., generated by a single element P say. Of course, we assume that E is non-empty. In this case we may actually compute the dimensions $N_d(E)$.

Lemma 2.1. $N_d(E)=N_d(\mathbb{R}^n)-N_{d-\deg(P)}(\mathbb{R}^n)$.

Proof. By our assumptions $\mathscr{P}_d(E)=\mathscr{P}_d/I(E)=\mathscr{P}_d/(P)$. Hence $p \sim q$ in $\mathscr{P}_d(E)$ iff $p-q=rP$ for some $r \in \mathscr{P}_{d-\deg(P)}$. It is not difficult to see that this implies that $N_d(E)=\dim(\mathscr{P}_d)-\dim(\mathscr{P}_{d-\deg(P)})$. ∎

Now suppose that $X \subset E$ is a unisolvent set of $N_d(E)$ points. In particular, if $p \in \mathscr{P}_d(E)$ is such that $p(x)=0$, $\forall x \in X$, then $p=0$ (in $\mathscr{P}_d(E)$). More generally, if $p \in \mathscr{P}_d$ is also such that $p(x)=0$, $\forall x \in X$, then $p \equiv 0$ on E; i.e., $p \in I(E)$ and so $p=rP$ for some $r \in \mathscr{P}$. We will have occasion to make use of these simple properties of unisolvent sets.

3. Configurations of Points Lying on a Collection of Algebraic Varieties

Suppose that $V_1, V_2, ..., V_m$ are algebraic varieties such that $I(V_i)$ is generated by $P_i \in \mathscr{P}_{d_i}$. We will also require that the V_i, pairwise, share no common components, or in other words, the P_i are pairwise relatively prime. We abbreviate $N_d(V_i)$ to N_d^i and $N_d(\mathbb{R}^n)$ to N_d. Our goal is to distribute N_d points over these m varieties in such a way as to produce a unisolvent set. However, the number of points which may be placed on each variety is limited.

LEMMA 3.1. *Suppose that $X \subset \mathbb{R}^n$ is a set of N_d points. If greater than N_d^i of these lie on V_i, then $VDM_{\mathbb{R}^n}^d(X) = 0$.*

Proof. We will show that then there is a $p \in \mathscr{P}_d$ such that $p(x) = 0$, $\forall x \in X$. By Lemma 2.1, $N_d^i = N_d - N_{d-d_i}$. Hence there would then be strictly fewer than N_{d-d_i} points not on V_i. Thus there would exist a $q \in \mathscr{P}_{d-d_i}$ which is zero at all the points off V_i. Then $p := P_i q$ is zero at all points of X. ∎

The numbers of points that we do place on each variety is based on the following simple calculation.

LEMMA 3.2. *Suppose that $d_1 + \cdots + d_{m-1} < d$ but that $d_1 + \cdots + d_m \geqslant d$. Then*

$$N_d^1 + N_{d-d_1}^2 + N_{d-d_1-d_2}^3 + \cdots + N_{d-d_1-\cdots-d_{m-1}}^m$$
$$= N_d - \begin{cases} 1 & \text{if } d_1 + \cdots + d_m = d \\ 0 & \text{if } d_1 + \cdots + d_m > d. \end{cases}$$

Proof. By Lemma 2.1, $N_d^1 + N_{d-d_1}^2 + \cdots + N_{d-d_1-\cdots-d_{m-1}}^m = \{N_d - N_{d-d_1}\} + \{N_{d-d_1} - N_{d-d_1-d_2}\} + \cdots + \{N_{d-d_1-\cdots-d_{m-1}} - N_{d-d_1-\cdots-d_m}\}$. If $d_1 + \cdots + d_m = d$, this collapses to $N_d - N_0 = N_d - 1$, and if $d_1 + \cdots + d_m > d$ to just N_d. ∎

We are now ready to describe the configurations of points which form the content of this note.

Configuration. Suppose that $V_1, ..., V_m$ are algebraic varieties as described above such that $d_1 + \cdots + d_{m-1} < d$ but $d_1 + \cdots + d_m \geqslant d$. Set $s_i = d - d_1 - \cdots - d_{i-1}$ and let $X_i \subset V_i$ be a set of $N_{s_i}^i$ distinct points, $i = 1, ..., m$. Further, suppose that the X_i are pairwise disjoint. Now, if $d_1 + \cdots + d_m > d$ set $X = X_1 \cup \cdots \cup X_m$ and if $d_1 + \cdots + d_m = d$, set $X = X_1 \cup \cdots \cup X_m \cup \{a\}$ where $a \in \mathbb{R}^n$ is not on any of the V_i.

By Lemma 3.2, $\operatorname{card}(X)=N_d$ and it is this set of points, X, which we consider. As it turns out, the unisolvency of X depends only on the (possibly) simpler problems of the unisolvency of the X_i.

THEOREM 3.3. *If* $VDM^{s_i}_{V_i}(X_i)\neq 0$, $i=1, ..., m$ *then* $VDM^d_{\mathbb{R}^n}(X)\neq 0$.

Proof. Suppose $p_1\in\mathscr{P}_d$ is such that $p_1(x)=0$, $\forall x\in X$. In particular, $p_1(x)=0$, $\forall x\in X_1$. But then, as $VDM^d_{V_1}(X_1)\neq 0$ by hypothesis, $p_1\in I(V_1)$ so that $p_1=p_2P_1$ for some $p_2\in\mathscr{P}_{d-d_1}$. Then $p_2(x)=0$, $\forall x\in X_2$, so that $p_2\in I(V_2)$ and hence $p_2=p_3P_2$ for some $p_3\in\mathscr{P}_{d-d_1-d_2}$. Continuing in this manner we see that $p_1=cP_1P_2\cdots P_m$ for some constant c. But if $d_1+\cdots+d_m>d$, as $\deg(p_1)=d$, this is impossible unless $c=0$. If $d_1+\cdots+d_m=d$ and $c\neq 0$, $p_1(a)=0$ implies that at least one of $P_1(a), ..., P_m(a)$ is zero, contradicting the choice of a. Hence $p_1=0$ and the result follows. ∎

In other words, X is unisolvent for interpolation by polynomials of degree d if each X_i is unisolvent for interpolation by polynomials of degree s_i on the variety V_i.

4. LAGRANGE POLYNOMIALS AND VANDERMONDE DETERMINANTS

If $X\subset\mathbb{R}^n$ is a unisolvent set of N_d points, then for each $x\in X$ we may form the Lagrange polynomial, $l_x\in\mathscr{P}_d$, defined by

$$l_x(y)=\delta_{x,y}\qquad\forall y\in X.$$

Here δ is the Kroneker delta. More generally, if $E\subset\mathbb{R}^n$ is some set and $X\subset E$ is unisolvent for interpolation by $\mathscr{P}_d(E)$, then again for each $x\in X$ we define the Lagrange polynomial $L_x\in\mathscr{P}_d(E)$ by

$$L_x(y)=\delta_{x,y}\qquad\forall y\in X.$$

The Lagrange polynomials are fundamental to the study of polynomial interpolation and much of the one-dimensional theory is based on an analysis of their properties. In several variables explicit formulas are only available for some simple cases, but as it turns out, there are recursive formulas for the points in our configurations.

THEOREM 4.1. *Suppose that* $X\subset\mathbb{R}^n$ *is a set of* N_d *points in the configuration of Section* 3 *which satisfies the unisolvency hypotheses of Theorem* 3.3. *For* $x\in X_i$ *let* $L_x\in\mathscr{P}_{s_i}(V_i)$ *be its associated Lagrange polynomial and* $\tilde{L}_x\in\mathscr{P}_{s_i}$

be such that $\tilde{L}_x|_{V_i} = L_x$. Then for $x \in X_i$, the Lagrange polynomial l_x is given by

$$l_x = \begin{cases} Q_x - \sum_{j=i+1}^{m} \sum_{y \in X_j} Q_x(y) l_y & \text{if } d_1 + \cdots + d_m > d \\ Q_x - \sum_{j=i+1}^{m} \sum_{y \in X_j} Q_x(y) l_y - Q_x(a) l_a & \text{if } d_1 + \cdots + d_m = d, \end{cases}$$

where

$$Q_x = \tilde{L}_x \times \prod_{j=1}^{i-1} \{P_j/P_j(x)\}.$$

In case $d_1 + \cdots + d_m = d$,

$$l_a = \prod_{j=1}^{m} \{P_j/P_j(a)\}.$$

Proof. First note that if $x \in X_i$ with $i > 1$, then $l_x(y) = 0$, $\forall y \in X_1$, and so $l_x \in I(V_1)$. Therefore, $l_x = q_1 P_1$ for some $q_1 \in \mathscr{P}_{d-d_1}$. Similarly, if $i > 2$, l_x is also zero on X_2 so that $q_1(y) = 0$, $\forall y \in X_2$. Hence $q_1 \in I(V_2)$ so that $q_1 = q_2 P_2$ for some $q_2 \in \mathscr{P}_{d-d_1-d_2}$. Continuing in this manner we see that $l_x|_{V_j} \equiv 0$ for $1 \leqslant j < i$.

Next consider Q_x. As $\tilde{L}_x \in \mathscr{P}_{s_i}$, $Q_x \in \mathscr{P}_d$. By construction $Q_x(y) = 0$, $\forall y \in X_j$, $1 \leqslant j < i$. Moreover, if $y \in X_i$, $y \neq x$, $\tilde{L}_x(y) = 0$, and so $Q_x(y) = 0$. Clearly $Q_x(x) = 1$. Hence Q_x has the correct values of l_x on X_j, $1 \leqslant j \leqslant i$. It needs only be adjusted to have zero values at X_j, $j > i$.

Now $\sum_{j=i+1}^{m} \sum_{y \in X_j} Q_x(y) l_y \in \mathscr{P}_d$, is zero on $V_1, ..., V_i$ by the first remark, and interpolates Q_x at $X_{i+1} \cup \cdots \cup X_m$. Thus subtracting this expression from Q_x has the desired effect. If $d_1 + \cdots + d_m = d$, then in addition $Q_x(a) l_a$ must also be subtracted. The formula for l_a is easily verified. ∎

The Vandermonde determinant is also of some interest. For the configurations of points considered here, it turns out that this determinant factors into the products of the smaller determinants belonging to X_i.

THEOREM 4.2. *Suppose that $X \subset \mathbb{R}^n$ is a set of points in the configuration of Section 3. Then*

$$VDM^d_{\mathbb{R}^n}(X) = C \cdot K \cdot \prod_{i=1}^{m} \left\{ \prod_{x \in X_i} \prod_{j=1}^{i-1} P_j(x) \right\} VDM^{s_i}_{V_i}(X_i),$$

where $C \neq 0$ is a constant independent of X and

$$K = \begin{cases} 1 & \text{if } d_1 + \cdots + d_m > d \\ \prod_{i=1}^{m} P_i(a) & \text{if } d_1 + \cdots + d_m = d. \end{cases}$$

Proof. Suppose first that $d_1 + \cdots + d_m > d$. Now the proof is much facilitated by the choice of a convenient basis for $\mathscr{P}_d$. So suppose that $B_i \subset \mathscr{P}_{s_i}$ is a set of $N^i_{s_i}$ polynomials such that $B_i|_{V_i}$ is a basis for $\mathscr{P}_{s_i}(V_i)$. Set $B = \bigcup_{i=1}^m \tilde{B}_i$, where $\tilde{B}_i = \{\prod_{j=1}^{i-1} P_j\} B_i$. We claim that B is a basis for $\mathscr{P}_d$. Clearly $B \subset \mathscr{P}_d$ and by Lemma 3.2, $\text{card}(B) = N_d$. Hence it suffices to show that B is a linearly independent set. But suppose that some linear combination $Q := \sum_{p \in B} \alpha_p p \equiv 0$. In particular, $Q|_{V_1} = 0$. But then, as for $i > 1$, each element of $\tilde{B}_i$ has a factor of P_1, $\sum_{p \in \tilde{B}_1} \alpha_p p = 0$ on V_1. By our choice of B_1 as a basis, $\alpha_p = 0$, $\forall p \in \tilde{B}_1$. Proceeding in a similar manner we see that $\alpha_p = 0$, $\forall p \in \tilde{B}_2$, etc. Hence all coefficients are 0 and B is indeed a basis.

Now form the Vandermonde matrix in the following block form. The first $N^1_{s_1}$ rows are all the basis polynomials evaluated at the points of X_1, the next $N^2_{s_2}$ rows are all the basis polynomials evaluated at the points of X_2, etc. Similarly, the first $N^1_{s_1}$ columns are the polynomials in $\tilde{B}_1$ evaluated at all the points, the next $N^2_{s_2}$ columns are those in $\tilde{B}_2$ evaluated at all the points, etc. By construction, the polynomials in $\tilde{B}_i$ are zero on X_j, $1 \leqslant j < i$, and so this matrix (Fig. 1) is actually block lower triangular and the determinant is the product of the determinants of the diagonal blocks.

Now consider the ith diagonal block. It is the matrix formed by evaluating the polynomials of $\tilde{B}_i$ at the points of X_i. But recall that $\tilde{B}_i = \{\prod_{j=1}^{i-1} P_j\} B_i$, where B_i restricts to a basis for $\mathscr{P}_{s_i}(V_i)$. Thus the row in this block belonging to $x \in X_i$ has the common factor, $\prod_{j=1}^{i-1} P_j(x)$ which may be factored out of the determinant. Hence the determinant of the ith block is $\{\prod_{x \in X_i} \prod_{j=1}^{i-1} P_j(x)\}\ VDM^{s_i}_{V_i}(X_i)$, from which the formula follows.

The case $d_1 + \cdots + d_m = d$ is similar and is omitted here. ∎

In an interesting special case we obtain a formula for the determinant highly reminiscent of the well-known one variable formula. Indeed, this case reduces to that formula when $n = 1$.

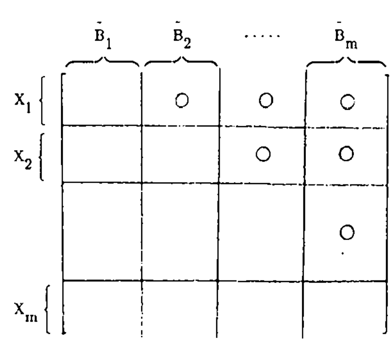

FIGURE 1

COROLLARY 4.3. *Suppose that $X \subset \mathbb{R}^n$ is a set of points in the configuration of Section 3, but that each of the varieties V_i are actually different level sets of a fixed polynomial. That is, there is a P such that $P_i = P - c_i$ for distinct constants c_i. Then*

$$VDM^d_{\mathbb{R}^n}(X) = C \cdot K \cdot \prod_{i=1}^{m} \left\{ \prod_{j=1}^{i-1} [c_i - c_j]^{N^i_{s_i}} \right\} VDM^{s_i}_{V_i}(X_i),$$

where $C \neq 0$ is a constant independent of X and

$$K = \begin{cases} 1 & \text{if } \; d_1 + \cdots + d_m > d \\ \prod_{i=1}^{m} \{P(a) - c_i\} & \text{if } \; d_1 + \cdots + d_m = d. \end{cases} \quad \blacksquare$$

Remark. C is independent of X but only under the condition that the V_i are fixed. C is dependent on the chosen basis of $\mathcal{P}_d$, and so in this last case, for instance, C *does* depend on the c_i. It is, however, guaranteed to be non-zero.

5. AN ILLUSTRATIVE EXAMPLE IN $\mathbb{R}^2$

Let $P(x_1, x_2) = x_1^2 + x_2^2$ and consider $P_i := P - R_i^2$ with distinct $R_i > 0$. V_i is then the circle of radius R_i, centred at the origin. In this case $d_1 = d_2 = \cdots = d_m = 2$ so that we must have $m = \lfloor (d+1)/2 \rfloor$. It is easily checked that $d_1 + \cdots + d_m > d$ when d is odd and equals d when d is even. In the even degree case we take our extra point $a := (0, 0)$. Then $s_i = d - 2(i-1)$ and as, in two variables, $N_d = (d+2)(d+1)/2$, the number of points placed on V_i is

$$N^i_{s_i} = N_{s_i} - N_{s_i - 2} = 2s_i + 1.$$

$\mathcal{P}_k(V_i)$ is the familiar space of trigonometric polynomials of degree k and as is well known, any $2k+1$ distinct points on the circle are unisolvent. Hence letting X_i consist of any $2s_i + 1$ distinct points on V_i (with one at the origin if d is even) always provides a unisolvent set. We begin by illustrating the calculation of the Lagrange polynomials. For simplicity we take $d = 2$ and $R_1 = 1$. Then $m = 1$, $N_2 = 6$, and $s_1 = 2(2) + 1 = 5$. Hence five points are placed on the unit circle and the sixth at the origin. Again, to simplify the calculation, suppose that the five on the unit circle are equally spaced with one at $(1, 0)$. We see easily that $l_{(0,0)}(x_1, x_2) = 1 - x_1^2 - x_2^2$ and calculate, for example, $l_{(1,0)}$ according to the formula of Theorem 4.1.

Restricted to the unit circle, we have a trigonometric interpolation problem and it is easily seen that

$$L_{(1,0)}(\theta) = \tfrac{2}{5}\{\tfrac{1}{2} + \cos(\theta) + \cos(2\theta)\}$$

so that

$$\tilde{L}_{(1,0)}(x_1, x_2) = \tfrac{2}{5}\{\tfrac{1}{2} + x_1 + (x_1^2 - x_2^2)\},$$

for instance. Hence, in this case,

$$Q_{(1,0)}(x_1, x_2) = \tfrac{2}{5}\{\tfrac{1}{2} + x_1 - x_1^2 - x_2^2\}$$

as well, and

$$\begin{aligned} l_{(1,0)}(x_1, x_2) &= Q_{(1,0)}(x_1, x_2) - Q_{(1,0)}(0, 0)\, l_{(0,0)}(x_1, x_2) \\ &= \tfrac{1}{5}\{2x_1 + 3x_1^2 - x_2^2\}. \end{aligned}$$

One form of the Vandermonde determinant may be immediately computed from Corollary 4.3. A curious consequence of this formula is that $V^d_{\mathbb{R}^n}(X)$ does not depend on the orientation of the points on X_i. This is due to the fact that the trigonometric Vandermondian is invariant under rotations.

Now in some circumstances it is useful to know precisely how the determinant depends on the points, or in other words, to know how the C depends on the radii. In order to see this we must compute the determinant using a basis for $\mathscr{P}_d$ which is independent of the P_i. It turns out that there is such an independent basis which is still convenient for the computation of the determinant. We give the construction for the d odd case. The d even case is only slightly different.

First let $\mathscr{B}_k := \{1, \mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Re}(z^2), \mathrm{Im}(z^2), ..., \mathrm{Im}(z^k)\}$, where $z := x_1 + ix_2$ and (temporarily) $i^2 = -1$. Then on the circle, $x_1^2 + x_2^2 = R^2$, $\mathscr{B}_k$ restricts to $\{1, R\cos(\theta), R\sin(\theta), ..., R^k \sin(k\theta)\}$ which is a basis for the trigonometric polynomials. Note also that $\mathscr{B}_{k-1} \subset \mathscr{B}_k$, $k = 1, 2, ...$ The existence of such "universal" bases is not particularly unusual.

LEMMA 5.1. *Let $P \in \mathscr{P}$ with $\deg(P) \geq 1$ and $c_1, c_2 \in \mathbb{R}$. Suppose that for $k = 0, 1, ..., d$, $\mathscr{B}_k \subset \mathscr{P}_k$ is such that $\mathscr{B}_k$ restricts to a basis for $\mathscr{P}_k/(P - c_1)$ and that $\mathscr{B}_{k-1} \subset \mathscr{B}_k$, $k = 1, ..., d$. Then $\mathscr{B}_d$ also restricts to a basis for $\mathscr{P}_d/(P - c_2)$.*

Proof. As the dimensions of $\mathscr{P}_d/(P - c_1)$ and $\mathscr{P}_d/(P - c_2)$ are the same we need only show linear independence of $\mathscr{B}_d$ in $\mathscr{P}_d/(P - c_2)$. Hence, suppose that $Q \in \mathrm{span}(\mathscr{B}_d)$ is such that $Q \sim 0$ in $\mathscr{P}_d/(P - c_2)$; i.e., $Q \in (P - c_2)$. Then $Q = (P - c_2)r$ for some $r \in \mathscr{P}_{d - \deg(P)}$, so that $Q = (P - c_1)r + (c_1 - c_2)r$. Thus, $Q \sim (c_1 - c_2)r$ in $\mathscr{P}_d/(P - c_1)$. But $\deg(r) = d - \deg(P)$ and so $Q \in \mathscr{B}_{d-\deg(P)}$ and hence $\deg(Q) = d - \deg(P)$. By repeating the argument, $\deg(Q) = d - 2\deg(P)$ and eventually we arrive at a contradiction. ∎

Now take $B_i := \mathscr{B}_{s_i}$ of above, $i = 1, ..., m$. Then $B_i|_{V_i}$ is a basis for $\mathscr{P}_{s_i}(V_i)$. Let $\tilde{D}_i := P^{i-1}B_i$ and set $D := \bigcup_{i=1}^m \tilde{D}_i$. In the proof of Theorem 4.2 we made use of the basis $B = \bigcup_{i=1}^m \tilde{B}_i$, where $\tilde{B}_i = \{\prod_{j=1}^{i-1} P_j\} B_i$. We claim that D is also a basis for $\mathscr{P}_d$ and, moreover, the Vandermonde determinants computed using B and D are *exactly* the same. This follows from the fact that the transition matrix from B to D has determinant $+1$. Again, a more general result is available.

LEMMA 5.2. *Suppose that $B_i \subset \mathscr{P}$, $i = 1, ..., m$ are such that $B_1 \supset B_2 \supset \cdots \supset B_m$ and that P is some polynomial. For $c_1, ..., c_m \in \mathbb{R}$ consider the two sets $B := \bigcup_{i=1}^m \{\prod_{j=1}^{i-1} (P - c_j)\} B_i$ and $D := \bigcup_{i=1}^m P^{i-1} B_i$. Then the transition matrix from B to D has determinant $+1$.*

Proof. Set $\tilde{B}_i := \{\prod_{j=1}^{i-1} (P - c_j)\} B_i$ and $\tilde{D}_i := P^{i-1} B_i$. Organize the transition matrix into the block structure shown in Fig. 2. Since $\tilde{B}_1 = \tilde{D}_1$, the first column is just $[I \mid 0 \mid 0 \mid \cdots \mid 0]'$. Now if $p \in \tilde{B}_2$, then $p = (P - c_1)q$ for some $q \in B_2$. But then $Pq \in \tilde{D}_2$ and as $B_2 \subset B_1 = \tilde{D}_1$, p can be expressed as a linear combination of the corresponding element in $\tilde{D}_2$ (i.e., Pq) and elements of $\tilde{D}_1$. Thus the second column is just $[* \mid I \mid 0 \mid \cdots \mid 0]'$. Continuing in this manner, it is not difficult to see that the matrix is block upper triangular with identities on the diagonal. Hence the determinant is $+1$. ∎

Using the bases B and D are therefore equivalent and so from Theorem 4.2,

$$VDM^d_{\mathbb{R}^n}(X) = C \cdot \prod_{i=1}^m \left\{ \prod_{x \in X_i} \prod_{j=1}^{i-1} P_j(x) \right\} VDM^{s_i}_{V_i}(X_i),$$

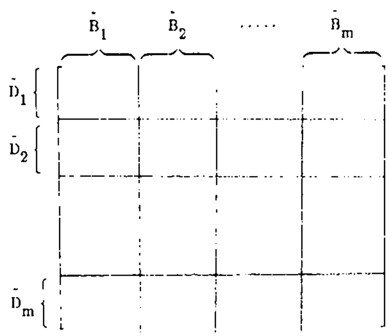

FIGURE 2

where $VDM^{s_i}_{V_i}(X_i)$ is computed using the basis $\mathscr{B}_{s_i}$ of above and C is completely independent of X. First note that we may simplify

$$\prod_{x\in X_i}\prod_{j=1}^{i-1} P_j(x)=\left\{\prod_{j=1}^{i-1}(R_i^2-R_j^2)\right\}^{2s_i+1}.$$

Second, if we write $X_i=\{R_i(\cos(\theta_j),\sin(\theta_j))\}_{j=1}^{2s_i+1}$ in trigonometric form,

$$VDM^{s_i}_{V_i}(X_i)=\begin{vmatrix} 1 & R_i\cos(\theta_1) & R_i\sin(\theta_1) & \cdots & R_i^{s_i}\sin(s_i\theta_1)\\ 1 & R_i\cos(\theta_2) & \cdots & & R_i^{s_i}\sin(s_i\theta_2)\\ \vdots & & & & \\ 1 & R_i\cos(\theta_{2s_i-1}) & R_i\sin(\theta_{2s_i+1}) & \cdots & R_i^{s_i}\sin(s_i\theta_{2s_i+1})\end{vmatrix}$$

which is just $R_i^{2\{1+2+\cdots+s_i\}}$ times an ordinary tirgonometric Vandermondian of degree s_i. Hence setting $\tilde{X}_i:=\{\theta_1,\ldots,\theta_{2s_i+1}\}$, we see that

$$VDM^d_{\mathbb{R}^n}(X)=C\cdot\prod_{i=1}^{m} R_i^{s_i(s_i+1)}\left\{\prod_{j=1}^{i-1}(R_i^2-R_j^2)\right\}^{2s_i+1} VDM^{s_i}_{x_1^2+x_2^2=1}(\tilde{X}_i),$$

where C is *completely* independent of X.

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